

# SPHERICAL FUNCTIONS ON SPHERES OF RANK TWO

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ABSTRACT. Under a suitable multiplicity freeness assumption we show that the spherical functions of the pair  $(\mathrm{SU}(n+1), \mathrm{SU}(n))$  can be described by families of vector-valued polynomials in two variables. The matrix weight that establishes the orthogonality is supported on the closed unit disc and depends on  $n$ . We calculate the matrix weight in a couple of examples and we show that in these examples the weight depends actually on  $n \in (1, \infty)$ .

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## 1. INTRODUCTION

Consider the Gelfand pair  $(U, L) = (\mathrm{SU}(n+1), \mathrm{SU}(n))$  with  $n \geq 2$  and an irreducible representation  $\pi^L : L \rightarrow \mathrm{GL}(V)$  with the following property: for every irreducible  $U$ -representation  $\pi^U : U \rightarrow \mathrm{GL}(V')$  the restriction  $\pi^U|_L$  contains  $\pi^L$  at most once, i.e.  $\dim \mathrm{Hom}_L(V, V') \leq 1$ .

1. We are interested in the space of functions  $F : U \rightarrow \mathrm{End}(V)$  such that

$$(1.1) \quad F(\ell_1 u \ell_2) = \pi^L(\ell_1) F(u) \pi^L(\ell_2) \quad \text{for all } (\ell_1, u, \ell_2) \in L \times U \times L.$$

For example, if  $\pi^L$  is the trivial representation, then the space of algebraic functions on  $U$  satisfying (1.1) can be given the structure of a polynomial algebra generated by two elements. The building blocks of this space are the zonal spherical functions, i.e. certain matrix coefficients of irreducible  $U$ -representations that contain an  $L$ -invariant vector. These functions form a vector space basis which is moreover orthogonal with respect to integration over  $U$  against the Haar measure (Schur orthogonality). Because the double cosets  $L \backslash U / L$  can be parametrized by the closed unit disc  $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$  one obtains families

of orthogonal polynomials in two variables. The orthogonality measure is given by  $(1 - x^2 - y^2)^{n-1} dx dy$ .

**1.1. Results.** In this note we show that under an additional assumption on the irreducible representation  $\pi^L$ , the space of functions  $F : U \rightarrow \text{End}(V)$  that satisfy (1.1) can be described as a space of vector-valued orthogonal polynomials in two variables. The additional condition amounts to requiring the existence of a face  $\mathcal{F}$  of the Weyl chamber of  $L$  such that the highest weight of  $\pi^L$  is in  $\mathcal{F}$  and the multiplicity freeness condition holds for all irreducible representations of  $L$  whose highest weight is contained in  $\mathcal{F}$ . Every face  $\mathcal{F}$  of the Weyl chamber of  $L$  of codimension one gives an example for this particular pair  $(U, L)$ .

The vector-valued polynomials that describe the spherical functions are orthogonal with respect to a matrix weight that is supported on the unit disc. We explain how to calculate the matrix weight in general. Because explicit calculations become rather involved we have used GAP [5] to obtain a couple of formulas. These formulas indicate that the matrix weight actually depends continuously on  $n$  for these examples.

**1.2. Related work.** If we take for  $\pi^L$  the trivial representation we obtain families of scalar-valued polynomials that have been investigated for example in [2]. The dependence on the parameter  $n$  has also been investigated. For these questions one brings differential operators into the game. The orthogonal polynomials that one obtains are related to Heckman-Opdam theory [20].

The pair  $(U, L)$  together with a face  $\mathcal{F}$  of the Weyl chamber of  $L$  for which the multiplicity freeness condition holds is called a multiplicity free system. These systems have been classified, see e.g. [15]. In the same paper the existence of families of multivariable vector-valued polynomials has been shown. Those families of polynomials, as well as the ones in this paper, have the property that they are determined as simultaneous eigenfunctions of a commutative algebra of differential operators. This is worked out systematically for multiplicity free systems related to compact symmetric pairs in [11].

The theory of multivariable vector-valued polynomials related to matrix coefficients on compact Gelfand pairs is a generalization of the theory of zonal spherical functions on compact symmetric spaces [6, Ch.5] and of vector-valued orthogonal polynomials related to compact Gelfand pairs of rank one [7].

**1.3. Future work.** The existence of the vector-valued polynomials is based on an *ad hoc* analysis of the spectrum of the induced representation  $\text{ind}_L^U \pi^L$ . The spectra for the other multiplicity free systems are currently being investigated by means of the theory of spherical varieties [13]. Once the existence (in the generality of the classification [15]) is established, many questions remain. For example, do shift operators exist and are the families of polynomials associated to different multiplicity free systems related by shift operators? The

answers are affirmative for particular classes of examples in the one-variable matrix valued cases [10, 19] and in the multivariable scalar-valued cases (Heckman-Opdam theory).

**1.4. Organization.** In Section 2 we introduce the zonal spherical functions on  $U$  and we show that it is sufficient to study their restrictions to a two dimensional torus  $A \subset U$  for which  $LAL = U$ . In Section 3 we use the theory of spherical varieties to obtain the decomposition of  $\text{ind}_L^U \pi^L$  under the appropriate multiplicity freeness assumption. We show that the spherical functions can be described by multivariable vector-valued orthogonal polynomials. In Section 4 we construct these polynomials and the matrix weight with respect to which they are orthogonal. In Section 5 we calculate a couple of examples using the GAP-code [16].

## 2. INVARIANT FUNCTIONS

Before we look at the invariant functions on  $U$  we establish some facts about the coset space  $L \backslash U / L$ .

**2.1. Spheres of rank two.** Consider  $\mathbb{C}^{n+1}$  with the standard basis  $(e_1, \dots, e_{n+1})$  equipped with the standard Hermitian inner product  $(v, w) = \sum_{i=1}^{n+1} \bar{v}_i w_i$ , where  $v = \sum_{i=1}^{n+1} v_i e_i$  and  $w = \sum_{i=1}^{n+1} w_i e_i$ . Let  $U = \text{SU}(n+1) \subset \text{GL}(\mathbb{C}^{n+1})$  denote the group of linear transformations with determinant one that leave the standard Hermitian form invariant, i.e.  $u \in U$  if and only if  $\det(u) = 1$  and for all  $v, w \in \mathbb{C}^{n+1}$  the equality  $(uv, uw) = (v, w)$  holds.

Let  $L \subset U$  be the stabilizer of  $e_{n+1}$  for the natural action  $U \times \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1} : (u, v) \mapsto uv$ . Then  $L$  is isomorphic to  $\text{SU}(n)$ . The image  $Ue_{n+1}$  is isomorphic to  $U/L \cong S^{2n+1} = \{v \in \mathbb{C}^{n+1} : (v, v) = 1\}$ . Define the elements

$$a_1(s) = \begin{pmatrix} e^{i(n-1)s} & 0 & 0 \\ 0 & e^{-2is} I_{n-1} & 0 \\ 0 & 0 & e^{i(n-1)s} \end{pmatrix}, \quad a_2(t) = \begin{pmatrix} \cos(t) & 0 & -\sin(t) \\ 0 & I_{n-1} & 0 \\ \sin(t) & 0 & \cos(t) \end{pmatrix}$$

in  $U$ . Furthermore, define

$$(2.1) \quad A_1 = \left\{ a_1(s) \left| \begin{array}{ll} s \in [0, \pi) & \text{if } n-1 \text{ even} \\ s \in [0, 2\pi) & \text{if } n-1 \text{ odd} \end{array} \right. \right\}$$

$$(2.2) \quad A_2 = \{a_2(t) | t \in [0, 2\pi)\}.$$

which are one-dimensional tori contained in  $U$ . Note that the two dimensional torus  $A = A_1 \cdot A_2$  is isomorphic to  $(A_1 \times A_2) / (A_1 \cap A_2)$ . Since  $A_1 \cap A_2 = \{1\}$  if  $n-1$  is even and  $A_1 \cap A_2 = \{1, a_2(\pi)\}$  if  $n-1$  is odd, we have

$$A = \{a_1(s)a_2(t) | (s, t) \in [0, \pi) \times [0, 2\pi)\}.$$

**Lemma 2.1.** *The map  $L \times A \times L \rightarrow U : (\ell, a, \ell') \mapsto \ell a \ell'$  is surjective.*

PROOF. First we show that the image of  $A$  under the orbit map  $U \rightarrow S^{2n+1} : u \mapsto ue_{n+1}$  parametrizes the  $L$ -orbits. Indeed, let  $v = v_1 e_1 + \dots + v_{n+1} e_{n+1} \in S^{2n+1}$  and write  $v'_1 = \sqrt{|v_1|^2 + \dots + |v_n|^2}$ . Use the  $L$ -action to rotate  $v$  to  $v'_1 e_1 + v_{n+1} e_{n+1}$ . There is a unique  $s \in [0, 2\pi/(n-1))$  such that  $v_{n+1} = e^{i(n-1)s} |v_{n+1}|$ . Use  $L$  once more to rotate  $v'_1 e_1 + v_{n+1} e_{n+1}$  to  $e^{i(n-1)s} (-|v'_1| e_1 + |v_{n+1}| e_{n+1})$ . There is a unique  $t \in [0, \pi/2]$  such that  $|v'_1| = \sin(t)$  and  $|v_{n+1}| = \cos(t)$ . Hence we can use  $L$  to rotate  $v$  to  $a_1(s) a_2(t) e_{n+1}$ .

It follows that  $L \times A \rightarrow U/L$  is surjective from which we deduce that  $L \times A \times L \rightarrow U$  is surjective.  $\square$

In view of Lemma 2.1 we call  $U/L$  a sphere of rank two. Note that the map  $A \rightarrow AL/L$  is generically  $2(n-1)$ -to-one if  $n-1$  is odd and  $n-1$ -to-one if  $n-1$  is even. We shall make this more precise in the next subsection

**2.2. Zonal Spherical Functions.** We identify irreducible representations of compact Lie groups with their highest weights according to the Theorem of the Highest Weight, see e.g. [8, Thm.5.110]. The set of dominant integral weights of  $U$  is denoted  $P_U^+$ . We have

$$P_U^+ = \mathbb{N}_0 \varpi_1 + \dots + \mathbb{N}_0 \varpi_n,$$

where the choice and the notation of the fundamental weights is taken from [8]. This means that  $\varpi_i$  is a character of the torus of diagonal elements  $T_U \subset U$  given by the formula  $\varpi_i(\text{diag}(t_1, \dots, t_{n+1})) = t_1 \cdots t_i$ .

The elements in  $P_U^+$  are the weights that can occur as the weight of a highest weight vector, so  $P_U^+$  classifies all irreducible representations modulo equivalence. Given an element  $\lambda \in P_U^+$  we denote the corresponding representation by  $\pi_\lambda^U : U \rightarrow \text{GL}(V_\lambda^U)$ . We equip every representation space  $V_\lambda^U$  with a Hermitian inner product  $\langle \cdot, \cdot \rangle_\lambda$  for which the  $U$ -representation is unitary. Note that  $\pi_0^U$  denotes the trivial  $U$ -representation.

We employ the same notation for the subgroup  $L$ , although we denote the fundamental weights by  $\omega_1, \dots, \omega_{n-1}$ . Furthermore, we denote  $T_L = T_U \cap L$ . Note that  $\varpi_i|_{T_L} = \omega_i$  for  $i = 1, \dots, n-1$  and  $\varpi_n|_{T_L} = 1$ .

**Definition 2.2.** Let  $P_{(U,L)}^+(0) = \{\lambda \in P_U^+ : [\pi_\lambda^U|_L : \pi_0^L] = 1\}$ . The set  $P_{(U,L)}^+(0)$  is called the 0-well.

The 0-well is a semigroup that is generated by  $\varpi_1, \varpi_n$ , see e.g. [12, Table 1]. Consider the left-right-regular representation of  $U \times U$  auf  $L^2(U, du)$ , where  $du$  is the normalized Haar measure. The space of invariants of this action restricted to  $L \times L$  is given by

$$L^2(U, du)^{L \times L} = \widehat{\bigoplus_{\lambda \in P_{(U,L)}^+(0)} \text{End}(V_\lambda^U)^{L \times L}},$$

where the hat indicates the Hilbert completion of the direct sum. The spaces  $\text{End}(V_\lambda^U)^{L \times L}$  are all one-dimensional since  $(U, L)$  is a Gelfand pair. Let  $v \in V_\lambda^U$  be an  $L$ -invariant vector of length one, i.e.  $\|v\| = 1$  and  $\pi_\lambda^U(\ell)v = v$  for all  $\ell \in L$ . The matrix coefficient

$u \mapsto \langle v, \pi_\lambda^U(u)v \rangle_\lambda$  is a zonal spherical function which we denote by  $\phi_\lambda : U \rightarrow \mathbb{C}$ . Furthermore, Schur-orthogonality [8, Cor.4.10] implies that  $\int_U \overline{\phi_\lambda(u)} \phi_{\lambda'}(u) du = \delta_{\lambda, \lambda'} / \dim(V_\lambda^U)$ . This shows that  $\{\phi_\lambda | \lambda \in P_{(U,L)}^+(0)\}$  is an orthogonal basis of  $L^2(U, du)^{L \times L}$ .

**Proposition 2.3.** *Let  $\lambda = \lambda_1 \varpi_1 + \lambda_2 \varpi_n$ . Then there exists  $c \in \mathbb{C}^\times$  with  $\phi_\lambda = c \phi_{\varpi_1}^{\lambda_1} \phi_{\varpi_n}^{\lambda_2} + \text{l.o.t.}$ , where a monomial  $\phi_{\varpi_1}^{\lambda'_1} \phi_{\varpi_n}^{\lambda'_2}$  is called of lower order if  $\lambda'_1 \leq \lambda_1$  or  $\lambda'_2 \leq \lambda_2$ .*

PROOF. Suppose that  $\lambda_1 > 1$ . The tensor product  $V_{\lambda - \varpi_1}^U \otimes V_{\varpi_1}^U$  decomposes into irreducible  $U$ -representations  $V_{\lambda'}^U$  with  $\lambda' \in \lambda - Q_{\text{sph}}^+$ , where  $Q_{\text{sph}}^+ = \mathbb{N}_0(\alpha_1 + \dots + \alpha_n) = \mathbb{N}_0(\varpi_1 + \varpi_n)$ . Here we use the theory of spherical roots, see e.g. [4, §4]. The spherical roots of the pair  $(\text{SL}(n+1, \mathbb{C}), \text{SL}(n, \mathbb{C}))$  are calculated in [3, §3(5)]. In terms of the Borel-Weil realizations of the representation spaces as sections of equivariant bundles, the Cartan projection  $V_{\lambda - \varpi_1}^U \otimes V_{\varpi_1}^U \rightarrow V_\lambda^U$  is given by the multiplication of algebraic functions. Hence  $\phi_{\lambda - \varpi_1} \phi_{\varpi_1} = \sum_{k=0}^{\min(\lambda_1, \lambda_n)} \tilde{c}_k \phi_{\lambda - k(\varpi_1 + \varpi_n)}$  with  $\tilde{c}_0 \neq 0$ . Moreover,  $\sum_{k=1}^{\min(\lambda_1, \lambda_n)} |\tilde{c}_k|^2 = 1$ .

The analogous statement holds for  $V_{\lambda - \varpi_n}^U \otimes V_{\varpi_n}^U$  if  $\lambda_2 > 0$  and the proof is completed by induction on  $\lambda_1 + \lambda_2$ .  $\square$

**Definition 2.4.** *Let  $\lambda \in P_{(G,L)}^+(0)$ . Then  $q_\lambda \in \mathbb{C}[z_1, z_2]$  denotes the unique polynomial for which  $\phi_\lambda = q_\lambda(\phi_{\varpi_1}, \phi_{\varpi_n})$ .*

Let  $\phi : A \rightarrow \mathbb{C}^2 : a \mapsto (\phi_{\varpi_1}(a), \phi_{\varpi_n}(a))$ . One readily verifies

$$\phi_{\varpi_1}(a_1(s)a_2(t)) = e^{i(n-1)s} \cos(t), \quad \phi_{\varpi_n}(a_1(s)a_2(t)) = e^{-i(n-1)s} \cos(t),$$

which shows that  $\phi : A \rightarrow \mathbb{C}^2$  maps into the real subspace spanned by  $g_1 = e_1 + e_2$  and  $g_2 = e_1 - e_2$  which is just the underlying real subspace of the diagonal in  $\mathbb{C}^2$ . The image  $\phi(A)$  is equal to the closed unit disc

$$(2.3) \quad D = \{xg_1 + yig_2 \mid x^2 + y^2 \leq 1\}.$$

Let  $h : S^{2n+1} \rightarrow \mathbb{C} : v \mapsto \langle v, e_1 \rangle$ ,  $d : \mathbb{C} \rightarrow \mathbb{C}^2 : z \mapsto z(e_1 + e_2)$  and let  $\gamma : A \rightarrow S^{2n+1} : a \mapsto ae_1$ . Then the diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & \mathbb{C}^2 \\ \gamma \downarrow & & \uparrow d \\ S^{2n+1} & \xrightarrow{h} & \mathbb{C} \end{array}$$

commutes. We mentioned at the end of the previous subsection that  $A \rightarrow AL/L$  is generically  $2(n-1)$ -to-one if  $n-1$  is odd and  $n-1$ -to-one if  $n-1$  is even. This is due to the fact that  $L \cap A$  has  $n-1$  and  $(n-1)/2$  elements depending on  $n-1$  odd or even. Besides, there is also an element in  $L$  that normalizes  $A$ . Indeed, the element  $\tilde{w} = \text{diag}(i, I_{n-1}, -i)$  conjugates  $a_2(t)$  to  $a_2(-t)$  and leaves  $A_1$  invariant. Conjugation with  $\tilde{w}.a_1(3\pi/(4(n-1))) \in L$  yields the same result. This conjugation generates the a group with two elements that we denote

by  $W$ . The actions of  $A \cap L$  and  $W$  commute, hence  $AL/L \cong A/((A \cap L) \times W)$ . Since the functions  $\phi_{\varpi_1}$  and  $\phi_{\varpi_2}$  restricted to  $A$  are invariant for the action of  $(A \cap L) \times W$ , they provide coordinates for  $AL/L$ , identifying  $AL/L$  with the closed disc. We describe the fibers of  $\phi$ .

- Let  $n - 1$  be odd. The fibers of  $\phi$  contain  $2(n - 1)$  elements when  $0 < x^2 + y^2 < 1$ ,  $n - 1$  elements when  $x^2 + y^2 = 1$  and  $\phi^{-1}(\{0\}) = A_1$ .
- Let  $n - 1$  be even. The fibers of  $\phi$  contain  $n - 1$  elements when  $0 < x^2 + y^2 < 1$ ,  $(n - 1)/2$  elements when  $x^2 + y^2 = 1$  and  $\phi^{-1}(\{0\}) = A_1$ .

It remains to simplify the integral of  $L$ -biinvariant functions over  $U$  to an integral over  $D$ . We invoke the discussion [2, §3.1] from which we deduce the equality

$$(2.4) \quad \int_U f(\phi(u))du = \frac{n}{\pi} \int_D f(z, \bar{z})(1 - |z|^2)^{n-1} d\lambda(z, \bar{z}), \quad f \in C(D)$$

where the disc  $D$  from (2.3) is identified with  $\{z \in \mathbb{C} : |z| \leq 1\}$  via  $d : \mathbb{C} \rightarrow \mathbb{C}^2$  and where  $d\lambda$  is the normalized Lebesgue-measure.

**Remark 2.5.** Let  $K \subset U$  be the image of the homomorphism  $U(n) \rightarrow SU(n + 1) : k \mapsto \text{diag}(k, \det k^{-1})$ . Then  $(U, K)$  is a compact Riemannian symmetric pair for which  $U = KA_2K$ . Harmonic analysis for such pairs greatly benefits from this decomposition and the rest of the well developed structure theory like the existence of restricted root systems. An important subgroup of  $K$  in this structure theory is  $M = Z_K(A_2)$  and we observe that  $M = A_1L_*$ , where  $L_*$  is the stabilizer of the  $L$ -action on  $S^{2n+1}$  for the point  $e_1$ . Note that  $L_*$  is isomorphic to  $SU(n - 1)$ .

The analysis of zonal spherical functions for a Riemannian symmetric pair like  $(U, K)$  restricted to a torus like  $A_1$  is closely related to the analysis of hypergeometric functions associated to root systems.

For compact Gelfand pairs there is not such a rich structure theory available. However, in this particular example, which is close to a Riemannian symmetric pair, we can also parametrize the double cosets by a torus. This is the motivation for stating the following result. Let  $da$  denote the normalized Haar measure on  $A$ .

**Proposition 2.6.** *Let  $f \in C(U)^{L \times L}$ . Then  $\int_U f(u)du = c \int_A f(a)|\delta_n(a)|da$ , where*

$$\delta_n(a_1(s)a_2(t)) = |\sin^2(t)|^{n-1} |\sin(2t)|$$

and  $c^{-1} = \int_A |\delta_n(a)|da$ .

PROOF. Let  $\mathfrak{a} = \text{Lie}(A)$  and consider  $\mathfrak{a} \rightarrow D : H \mapsto \phi(\exp(H))$ . With respect to the coordinates  $(s, t)$  on  $\mathfrak{a}$  for which  $\exp(s, t) = a_1(s)a_2(t)$  the Jacobian is given by a multiple of  $(s, t) \mapsto \sin(2t)$ . In view of (2.4) this implies the claim.  $\square$

**2.3. Spherical functions.** Let  $\pi_\mu^L$  be an irreducible representation that induces multiplicity free to  $U$ , i.e.  $\mu \in P_L^+$  is such that for all  $\lambda \in P_U^+$  we have  $\dim \text{Hom}_L(V_\mu^L, V_\lambda^U) \leq 1$ . In this case we call  $(U, L, \mu)$  a multiplicity free triple.

Let  $F$  be a face of the Weyl chamber of  $L$  such that for all elements  $\mu \in P_L^+ \cap F$  the representation  $\pi_\mu^L$  induces multiplicity free to  $U$ . In that case we call the triple  $(U, L, F)$  a multiplicity free system.

Since  $(U, L)$  is a Gelfand pair,  $(U, L, \{0\})$  is a multiplicity free system. In Section 3 we investigate the induction of  $\pi_\mu^L$  to  $U$  in greater detail and we shall see that there are more examples of multiplicity free systems other than only  $(U, L, \{0\})$ . In this section we investigate the spherical functions up to the point where we need this induction. Throughout the rest of this section we fix  $\mu \in P_L^+$  such that  $(U, L, \mu)$  is a multiplicity free triple.

**Definition 2.7.** *The set  $P_{(U,L)}^+(\mu) = \{\lambda \in P_U^+ \mid [\pi_\lambda^U|_L : \pi_\mu^L] = 1\}$  is called the  $\mu$ -well.*

Consider the  $L \times L$ -action on  $\text{End}(V_\mu^L)$  via  $((\ell_1, \ell_2), x) \mapsto \pi_\mu^L(\ell) \circ x \circ \pi_\mu^L(\ell_2)^{-1}$ . The inner product  $(x, y) \mapsto \text{tr}(x^*y)$  on  $\text{End}(V_\mu^L)$ , where  $x^*$  denotes the Hermitian adjoint with respect to the invariant Hermitian inner product on  $V_\mu^L$ , is  $L \times L$ -invariant. Together with the left-right-regular representation of  $L \times L$  in  $L^2(U, du)$  we obtain the  $L \times L$ -representation in the tensor product

$$L^2(U, du) \otimes \text{End}(V_\mu^L),$$

which is unitary for the Hermitian inner product given by

$$(2.5) \quad \langle \Phi, \Psi \rangle_\mu = \int_U \text{tr}(\Phi(u)^* \Psi(u)) du.$$

The space of  $L \times L$ -invariants decomposes as the Hilbert completion

$$(2.6) \quad (L^2(U, du) \otimes \text{End}(V_\mu^L))^{L \times L} = \widehat{\bigoplus}_{\lambda \in P_{(U,L)}^+(\mu)} \text{Hom}_L(V_\mu^L, V_\lambda^U) \otimes \text{Hom}_L(V_\lambda^U, V_\mu^L).$$

The spaces  $\text{Hom}_L(V_\mu^L, V_\lambda^U) \otimes \text{Hom}_L(V_\lambda^U, V_\mu^L)$  are one dimensional by assumption. Let  $j : V_\mu^L \rightarrow V_\lambda^U$  be an isometric  $L$ -invariant linear map and let  $p : V_\lambda^U \rightarrow V_\mu^L$  denote its adjoint. Then

$$\Phi_\lambda^\mu : U \rightarrow \text{End}(V_\mu^L) : u \mapsto p \circ \pi_\lambda^U \circ j$$

is called a spherical function of type  $\mu$  associated to  $\lambda$ . In fact,  $\Phi_\lambda^\mu \in \text{Hom}_L(V_\mu^L, V_\lambda^U) \otimes \text{Hom}_L(V_\lambda^U, V_\mu^L)$  in the decomposition (2.6). Schur orthogonality implies

$$(2.7) \quad \langle \Phi_\lambda^\mu, \Phi_{\lambda'}^\mu \rangle_\mu = \frac{\dim(V_\mu^L)^2}{\dim(V_\lambda^U)} \delta_{\lambda, \lambda'},$$

see e.g. [14, Prop.3.3.25]. According to the decomposition (2.6) the set  $\{\Phi_\lambda^\mu \mid \lambda \in P_{(U,L)}^+(\mu)\}$  is an orthogonal basis of  $(L^2(U, du) \otimes \text{End}(V_\mu^L))^{L \times L}$ .

Finally we note that in view of the transformation behavior of  $\Psi \in (L^2(U, du) \otimes \text{End}(V_\mu^L))^{L \times L}$ , given by

$$\Psi(\ell_1 u \ell_2) = \pi_\mu^L(\ell_1) \Psi(u) \pi_\mu^L(\ell_2) \quad \text{for all } (\ell_1, \ell_2, u) \in L \times L \times U,$$

it is enough to know the restrictions  $\Psi|_A$ . Let  $L_* \cong \text{SU}(n-1)$  be the isotropy group of  $e_1 \in S^{2n+1}$  for the action of  $L$  on  $S^{2n+1}$ . Then  $L_*$  commutes with  $A$ . We see that  $\Psi(A) \subset \text{End}_{L_*}(V_\mu^L)$ . Let  $v \in S^{2n+1} \setminus \{e_{n+1}\}$ . Then the stabilizer  $L_v$  is conjugated to  $L_*$ . This implies that the complexification of  $L_*$  is equal to the generic stabilizer of the action of the complexification  $L_{\mathbb{C}}$  of  $L$  on the annihilator  $\mathfrak{t}_{\mathbb{C}}^\perp = \{f \in \mathfrak{u}_{\mathbb{C}}^* \mid \forall X \in \mathfrak{t}_{\mathbb{C}} : f(X) = 0\}$  of  $\mathfrak{t}_{\mathbb{C}}$  in  $\mathfrak{u}_{\mathbb{C}}^*$ . The restriction  $\pi_\mu^L|_{L_*}$  decomposes multiplicity free, see e.g. [15, Prop.2.4]. It follows that the elements  $\Psi(a), a \in A$  can be diagonalized simultaneously.

For later reference we record the following remark.

**Remark 2.8.** The integrand in (2.5) is  $L \times L$ -invariant. This implies that the inner product (2.5) can be given as an integration of  $A$ ,

$$\langle \Phi, \Psi \rangle_\mu = c \int_A \Psi(a)^* \Phi(a) |\delta_n(a)| da,$$

where we have used Proposition 2.6.

### 3. MULTIPLICITY FREE INDUCTION

Let  $i \in \{1, \dots, n-1\}$  and let  $\mu \in P_L^+$  be of the form  $\mu = \sum_{j=1}^{n-1} \mu_j \omega_j$  with  $\mu_i = 0$  and write  $\lambda = \sum_{j=1}^n \lambda_j \varpi_j$ . We want to describe the irreducible  $U$ -representations that contain  $\pi_\mu^L$  upon restriction to  $L$ . Instead of the classical branching rules adapted to our situation we apply the theory of spherical varieties, in particular the extended weight semigroup (see e.g. [1]). To explain this object we consider the complexifications  $G = \text{SL}(n+1, \mathbb{C})$  and  $H = \text{SL}(n, \mathbb{C})$  of the compact groups  $U$  and  $L$  respectively. The roots and choices for positivity are essentially the same for  $U$  and  $G$  and for  $L$  and  $H$ . Using Weyl's unitary trick we employ the identifications  $P_U^+ = P_G^+$  and  $P_L^+ = P_H^+$ . In the same spirit we define and identify  $P_{(G,H)}^+(\mu) = P_{(U,L)}^+(\mu)$  for all  $\mu \in P_H^+$ .

Let  $P_{\alpha_i} \subset H$  be the standard parabolic subgroup of  $H$  associated to the positive root  $\alpha_i$ . This means that the semisimple part of  $P_{\alpha_i}$  is isomorphic to  $\text{SL}(2, \mathbb{C})$  with positive root  $\alpha_i$ . The character  $\mu : T_L \rightarrow \mathbb{C}^\times$  can be extended to a character  $\mu : P_{\alpha_i} \rightarrow \mathbb{C}^\times$ . Let  $H \times^{P_{\alpha_i}} \mathbb{C}$  denote the associated line bundle over  $H/P_{\alpha_i}$ . The representation  $\pi_\mu^H$  can be realized in the space of sections of the vector bundle  $H \times^{P_{\alpha_i}} \mathbb{C}$  over  $H/P_{\alpha_i}$ .

It is known (see [15, Thm.2.1]) that  $P_{\alpha_i} \subset G$  is a spherical subgroup, i.e.  $G$  contains a Borel subgroup that admits an open orbit in the quotient  $G/P_{\alpha_i}$ . This is equivalent to the fact that the spaces of sections of all  $G$ -equivariant line bundles over  $G/P_{\alpha_i}$  decompose multiplicity free into direct sums of irreducible  $G$ -representations, see [21, Thm.25.1]. Let  $X^*(P_{\alpha_i})$  denote the group of characters of  $P_{\alpha_i}$ .



**Definition 3.1.** The extended weight semigroup  $\widehat{\Lambda}(G, P_{\alpha_i}) \subset P_U^+ \times X^*(P_{\alpha_i})$  consists of the pairs  $(\lambda, \chi)$  such that  $V_\lambda^G$  contains a line on which  $P_{\alpha_i}$  acts with character  $-\chi$ .

It turns out that  $\widehat{\Lambda}(G, P_{\alpha_i})$  is a finitely generated free semigroup, see [1, Thm.2]. Using induction in stages and Frobenius reciprocity one shows:  $\lambda \in P_{(G,H)}^+(\mu)$  if and only if  $(\lambda^*, \mu) \in \widehat{\Lambda}(G, P_{\alpha_i})$ , where  $\lambda^*$  is the highest weight of the representation dual to  $V_\lambda^G$ .

**Lemma 3.2.** Let  $i \in \{1, \dots, n-1\}$ . The extended weight semigroup  $\widehat{\Lambda}(G, P_{\alpha_i})$  is freely generated by the  $2n-2$  elements

$$(3.1) \quad (\varpi_j^*, \omega_j) \quad \text{with } 1 \leq j \leq n \text{ and } j \neq i,$$

$$(3.2) \quad (\varpi_j^*, \omega_{j-1}) \quad \text{with } 1 \leq j \leq n \text{ and } j \neq i+1.$$

PROOF. Let  $\mathrm{GL}(n, \mathbb{C}) \subset G$  denote the image of the map  $x \mapsto \mathrm{diag}(x, \det x^{-1})$ . The standard Borel subgroup  $B_{\mathrm{GL}(n, \mathbb{C})} \subset \mathrm{GL}(n, \mathbb{C})$  consisting of upper-triangular matrices remains spherical in  $G$ . The extended weight semigroup  $\widehat{\Lambda}(G, B_{\mathrm{GL}(n, \mathbb{C})})$  has been calculated in [17, L.2.2]. Its generators are

$$(3.3) \quad (\varpi_j^*, \varpi_j), \quad (\varpi_j^*, \varpi_{j-1} - \varpi_n), \quad j = 1, \dots, n.$$

An element of  $\widehat{\Lambda}(G, P_{\alpha_i})$  can be viewed as an element of  $\widehat{\Lambda}(G, B_{\mathrm{GL}(n, \mathbb{C})})$  by extending the second component to the center of  $\mathrm{GL}(n, \mathbb{C})$  trivially. As such it can be written as a linear combination of the elements (3.3) where the coefficients are in  $\mathbb{N}_0$ . Certainly the coefficients of  $(\varpi_i^*, \varpi_i)$  and  $(\varpi_{i+1}^*, \varpi_i - \varpi_n)$  are zero. This shows that the elements (3.1),(3.2) generate  $\widehat{\Lambda}(G, P_{\alpha_i})$ . Since these elements are also linearly independent, the result follows.  $\square$

Let  $i \in \{1, \dots, n-1\}$  and let  $\mu \in P_L^+$  be of the form  $\mu = \sum_{j=1}^{n-1} \mu_j \varpi_j$  with  $\mu_i = 0$  and write  $\lambda = \sum_{j=1}^n \lambda_j \varpi_j$ . Then  $(\lambda^*, \mu) \in \widehat{\Lambda}(G, P_{\alpha_i})$  if and only if there are  $(r, s) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$  with

- $r_j + s_j = \lambda_j$  for  $j = 1, \dots, n$ ,
- $r_j + s_{j+1} = \mu_j$  for  $j = 1, \dots, n-1$ .

Note that  $r_i = 0, s_{i+1} = 0$  automatically. We say that the pair  $(r, s) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$  determines the pair  $(\lambda^*, \mu)$  and vice versa.

If  $\lambda \in P_{(G,H)}^+(\mu)$  so that  $(\lambda^*, \mu)$  is determined by the pair  $(r, s) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ , then  $\lambda - r_n \varpi_n - s_1 \varpi_1 \in P_{(G,H)}^+(\mu)$ . This inspires the following definition,

$$B(\mu) := \{\lambda \in P_{(G,H)}^+(\mu) \mid r_n = s_1 = 0\},$$

called the *bottom of the  $\mu$ -well*. Moreover, we define the  $\mu$ -degree  $d_\mu(\lambda) := (r_n, s_1)$  and the total  $\mu$ -degree  $|d_\mu(\lambda)| = r_n + s_1$ . We denote  $\nu_\lambda = \lambda - r_n \varpi_n - s_1 \varpi_1 \in B(\mu)$ .

**Theorem 3.3.** Let  $i \in \{1, \dots, n-1\}$  and let  $\mu \in P_H^+$  be of the form  $\mu = \sum_{j=1}^{n-1} \mu_j \omega_j$  with  $\mu_i = 0$ . Then

$$P_{(G,H)}^+(\mu) = B_{(G,H)}(\mu) + P_{(G,H)}^+(0).$$

PROOF. Immediate from the preceding discussion by writing  $\lambda = \lambda - r_n \varpi_n - s_1 \varpi_1 + (r_n \varpi_n + s_1 \varpi_1)$ .  $\square$

Let  $H_*$  denote the complexification of  $L_*$ . Let  $T_{H_*} \subset H_*$  denote the maximal torus of  $H_*$  consisting of diagonal elements. Then  $A_{\mathbb{C}} T_{H_*} \subset G$  is a maximal torus of  $G$ , where  $A_{\mathbb{C}}$  is the complexification of  $A$ . Fix a Weyl chamber for this torus that projects to the positive Weyl chamber of  $H_*$  determined by the upper triangular matrices. Let  $\tilde{\lambda}$  be a dominant integral weight and suppose that the corresponding irreducible representation  $\pi_{\tilde{\lambda}}^G$  contains  $\pi_{\mu}^H$  upon restriction to  $H$ . The restriction  $\tilde{\lambda}_*$  of  $\tilde{\lambda}$  to  $T_{H_*}$  is also dominant and integral. The corresponding irreducible representation  $\pi_{\tilde{\lambda}_*}^{H_*}$  of  $H_*$  occurs with multiplicity one in the restriction  $\pi_{\mu}^H|_{H_*}$ . This sets up a bijection

$$B(\mu) \rightarrow \{\sigma \in P_{H_*}^+ \mid \dim \text{Hom}_{H_*}(V_{\sigma}^{H_*}, V_{\mu}^H) \geq 1\},$$

see [15, Thm.3.1]. The torus  $A_{\mathbb{C}} T_{H_*}$  is conjugated to  $T_G$  and projection along  $A_{\mathbb{C}}$  corresponds to projecting along  $\varpi_1$  and  $\varpi_n$ .

**Definition 3.4.** *The elements of  $B(\mu)$  can be regarded as the highest weights of the irreducible constituents of  $\pi_{\mu}^H|_{H_*}$ . As such, the element  $\nu \in B(\mu)$  is denoted by  $\nu_*$ .*

**Proposition 3.5.** *Let  $i \in \{1, \dots, n-1\}$  and let  $\mu \in P_H^+$  be of the form  $\mu = \sum_{j=1}^{n-1} \mu_j \omega_j$  with  $\mu_i = 0$ . Let  $\lambda, \lambda' \in P_{(G,H)}^+(\mu)$ . If  $\dim \text{Hom}_G(V_{\lambda'}^G, V_{\lambda}^G \otimes V_{\varpi_i}^G) \geq 1$  for  $i = 1$  or  $i = n$ , then  $|d_{\mu}(\lambda')| \leq |d_{\mu}(\lambda)|$ .*

PROOF. It is enough to show that  $|d_{\mu}(\lambda - \alpha_j)| \leq |d_{\mu}(\lambda)|$  for all  $j = 1, \dots, n$ , whenever  $\lambda - \alpha_j \in P_{(G,H)}^+(\mu)$ . Note that  $\lambda - \alpha_i \notin P_{(G,H)}^+(\mu)$ . In general we have

- $(\alpha_1, 0) = (\varpi_1, \varpi_1) + (\varpi_1, 0) - (\varpi_2, \varpi_1)$ ,
- $(\alpha_j, 0) = -(\varpi_{j-1}, \varpi_{j-1}) + (\varpi_j, \varpi_j) + (\varpi_j, \varpi_{j-1}) - (\varpi_{j+1}, \varpi_j)$ , for  $j = 2, \dots, n-1$ ,
- $(\alpha_n, 0) = -(\varpi_{n-1}, \varpi_{n-1}) + (\varpi_n, \varpi_{n-1}) + (\varpi_n, 0)$ ,

which shows that either the  $\mu$ -degree does not change or the total  $\mu$ -degree drops by one after subtracting a simple root.  $\square$

**Corollary 3.6.** *Let  $i \in \{1, \dots, n-1\}$  and let  $\mu \in P_H^+$  be of the form  $\mu = \sum_{j=1}^{n-1} \mu_j \omega_j$  with  $\mu_i = 0$  and let  $\lambda \in P_{(G,H)}^+(\mu)$  be associated to  $(r, s) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ . Then there exist polynomials  $q_{\lambda, \nu}^{\mu} \in \mathbb{C}[\phi_{\varpi_1}, \phi_{\varpi_n}]$ ,  $\nu \in B(\mu)$ , such that*

$$(3.4) \quad \Phi_{\lambda}^{\mu}(g) = \sum_{\nu \in B(\mu)} q_{\lambda, \nu}^{\mu}(\phi_{\varpi_1}(g), \phi_{\varpi_n}(g)) \Phi_{\nu}^{\mu}(g).$$

*The polynomials are uniquely determined by  $\Phi_{\lambda}^{\mu}$ . Moreover, the total degree of  $q_{\lambda, \nu}^{\mu}$  is  $r_n + s_1$ . The total degrees of all other polynomial coefficients in (3.4) are of total degree  $\leq r_n + s_1$ . If the degree  $(r'_n, s'_1)$  of a polynomial coefficient  $q_{\lambda, \nu}^{\mu}$  in (3.4) satisfies  $r_n + s_1 = r'_n + s'_1$  then  $\nu + r_n \varpi_n + s_1 \varpi_1 \leq \lambda$  in the usual ordering.*

PROOF. The proof is with induction on  $r_n + s_1$ , similar to the proof of Proposition 2.3. In view of Theorem 3.3 and Proposition 3.5 we only have to show that the coefficient  $c_i$  in  $\phi_{\varpi_i} \Phi_\lambda^\mu = c_i \Phi_{\lambda+\varpi_i}^\mu + \text{l.o.t.}$  is non-zero for  $i = 1, n$ . This follows from surjectivity of the Cartan projection  $V_\lambda^U \otimes V_{\varpi_i}^U \rightarrow V_{\lambda+\varpi_i}^U$ .  $\square$

#### 4. VECTOR-VALUED ORTHOGONAL POLYNOMIALS

Let  $i \in \{1, \dots, n-1\}$  and let  $\mu \in P_H^+$  be of the form  $\mu = \sum_{j=1}^{n-1} \mu_j \omega_j$  with  $\mu_i = 0$ . Fix a total ordering on  $B(\mu) = \{\nu_1, \dots, \nu_N\}$  that is compatible with the usual partial ordering on  $P_G^+$ . The space of  $L_*$ -equivariant endomorphisms  $\text{End}_{L_*}(V_\mu^L)$  consists of block matrices, the blocks being multiples of the identity that correspond to the  $L_*$ -intertwiner of the  $L_*$ -isotypical type  $\nu_{i,*}$ . These multiples are collected in a vector in  $\mathbb{C}^N$ . The linear map  $i : \text{End}_{L_*}(V_\mu^L) \rightarrow \mathbb{C}^N$  that we obtain in this way is unitary for the Hermitian inner products  $(x, y) \mapsto \text{tr}(x^*y)$  and  $(z, \zeta) \mapsto z^* T^\mu \zeta$ , where  $T^\mu$  is the diagonal matrix whose entries are the dimensions of  $V_{\nu_{i,*}}^L$ . We identify the space of invariant functions (2.6) restricted to  $A$  with  $\mathbb{C}^N$ -valued functions on  $A$ . In particular we denote

$$\Psi_\lambda^\mu(a) = i(\Phi_\lambda^\mu(a)).$$

**Definition 4.1.** (1) Let  $\Psi_0^\mu : A \rightarrow \text{Mat}(\mathbb{C}, N \times N)$  be the matrix-valued function on  $A$  whose  $i$ -th column in  $a$  is given by  $\Psi_{\nu_i}^\mu(a)$ .

(2) Let  $\widetilde{W}_{\text{pol}}^\mu : A \rightarrow \text{Mat}(\mathbb{C}, N \times N)$  be the matrix-valued function on  $A$  defined by  $\widetilde{W}_{\text{pol}}^\mu(a) := \Psi_0^\mu(a)^* T^\mu \Psi_0^\mu(a)$ .

**Lemma 4.2.** The function  $\widetilde{W}_{\text{pol}}^\mu$  is positive definite almost everywhere on  $A$ .

PROOF. Since  $T^\mu$  has real entries,  $\widetilde{W}_{\text{pol}}^\mu$  is certainly self-adjoint. By [15, L.6.1],  $\Psi_0^\mu$  is invertible on a dense open subset of  $A$ , and the result follows.  $\square$

We define  $Q_\lambda^\mu \in \mathbb{C}^N[x, y]$  by  $\Psi_0^\mu Q_\lambda^\mu(\phi_{\varpi_1}, \phi_{\varpi_n}) = \Psi_\lambda^\mu$ , which makes sense because  $\Psi_0^\mu$  is invertible almost everywhere.

**Lemma 4.3.** The entries of the function  $\widetilde{W}_{\text{pol}}^\mu$  are polynomials in  $\phi_{\varpi_1}, \phi_{\varpi_n}$ .

PROOF. The  $(i, j)$ -entry of  $\widetilde{W}_{\text{pol}}^\mu(a)$  is given by  $\text{tr}(\Phi_{\nu_i}^\mu(a)^* \Phi_{\nu_j}^\mu(a))$ , which is the restriction of an  $L \times L$  invariant sum of matrix coefficients on  $U$  to  $A$ , hence it can be written as a polynomial in  $\phi_{\varpi_1}, \phi_{\varpi_n}$ .  $\square$

**Definition 4.4.** Let  $W_{\text{pol}}^\mu \in \text{Mat}(\mathbb{C}, N \times N)[z, \bar{z}]$  be defined by  $W_{\text{pol}}^\mu(\phi_{\varpi_1}(a), \phi_{\varpi_n}(a)) = \widetilde{W}_{\text{pol}}^\mu(a)$  and  $w_n \in C(\mathbb{C}^2)$  by  $w_n(z, \bar{z}) = (1 - |z|^2)^{n-1}$ .

Let  $Q_1, Q_2 \in \mathbb{C}^N[z, \bar{z}]$  and consider the pairing

$$(Q_1, Q_2) \mapsto \langle Q_1, Q_2 \rangle_\mu = \int_D Q_1^\mu(z, \bar{z})^* W_{\text{pol}}^\mu(z, \bar{z}) Q_2^\mu(z, \bar{z}) w_n(z, \bar{z}) d\lambda(z).$$

This provides  $\mathbb{C}^N[z, \bar{z}]$  with a Hermitian inner product. Let  $\overline{\mathbb{C}^N[z, \bar{z}]^\mu}$  denote its completion.

**Theorem 4.5.** *The map  $\Phi_\lambda^\mu \mapsto Q_\lambda^\mu$  extends to an isometry*

$$(L^2(U, du) \otimes \text{End}(V_\mu^L))^{L \times L} \rightarrow \overline{\mathbb{C}^N[x, y]}^\mu.$$

PROOF. This is clear from the preceding discussion.  $\square$

It follows that the space spanned by the spherical functions of type  $\mu$  can be identified with a space of vector-valued orthogonal polynomials in two variables as long as  $\mu$  satisfies the multiplicity freeness assumption. The spherical functions are also determined as simultaneous eigenfunctions of a commutative subquotient of  $U(\mathfrak{g})$ , see [11, L.2.2]. This implies that the vector-valued orthogonal polynomials are also determined as simultaneous eigenfunctions of a commutative algebra of differential operators. In the case  $\mu = 0$  we have control over this algebra by means of the Harish-Chandra homomorphism for spherical varieties, see [9]. In the general case this algebra is not yet fully understood.

## 5. EXAMPLES

In this section we are concerned with explicit expressions of the functions  $\Psi_0^\mu : A \rightarrow \text{Mat}(\mathbb{C}, N \times N)$ , as these functions determine the polynomial part of the matrix weight. The entries of  $\Psi_0^\mu$  are matrix coefficients of  $U$ -representations that are restricted to  $A$ . More precisely, the  $(i, j)$ -th entry is the matrix coefficient

$$a \mapsto m_{v_{\nu_{i,*}}, v_{\nu_{j,*}}}^{\nu_j}(a),$$

where  $v_{\nu_{i,*}} \in V_\mu^L \subset V_{\nu_j}^U$  is a highest weight vector of  $L_*$  of weight  $\nu_{i,*}$ . These calculations can be done systematically by computer algebra. Indeed, in [18, §4.1] such an algorithm is provided for a different pair of groups. We have adapted this algorithm to our situation. First we show that we only have to calculate the restriction  $(\Psi_{\nu_i}^\mu)|_{A_1}$  because of the transformation behavior of  $\Psi_{\nu_i}^\mu$  under  $A_2$ .

**Lemma 5.1.** *There exist integers  $m_{i,j}, 1 \leq i, j \leq N$  such that*

$$\Psi_{\nu_i}^\mu(a_2(s)a) = \text{diag}(e^{im_{i,1}s}, \dots, e^{im_{i,N}s})\Psi_{\nu_i}^\mu(a)$$

for  $a_2(s) \in A_2$  and  $a \in A$ .

PROOF. Let  $\lambda \in P_{(U,L)}^+(\mu)$  be associated to the pair  $(r, s) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ . The pair  $(r, s)$  corresponds to an element  $(\lambda, \tilde{\mu}) \in \widehat{\Lambda}(G, B_{\text{GL}(n, \mathbb{C})})$ . The  $\text{GL}(n, \mathbb{C})$ -module of highest weight  $\tilde{\mu}$  restricted to  $L \subset \text{GL}(n, \mathbb{C})$  is just  $V_\mu^L$ . Let  $M = L_* \cdot A_2$ . The  $M_{\mathbb{C}}$ -subrepresentations of  $\pi_{\tilde{\mu}}^{\text{GL}(n, \mathbb{C})}$  restricted to  $L_*$  are clearly  $A_2$ -stable. Hence  $A_2$  acts with a character on the  $L_*$ -subrepresentations of  $V_\mu^L$ .  $\square$

For the actual computations we have restricted ourselves to a couple of examples in a small class of representations of  $L$ , namely those of highest weight  $k\omega_{n-1}$ . The reason is that this greatly simplifies the determination of the highest weight vectors of the irreducible  $L_*$ -representations of  $V_\mu^L$ , which is step 7 of the algorithm in [19, §4.1]. Let  $E_{i,j} \in \mathfrak{g}$  denote the

element with zeros everywhere except for a one in the entry  $(i, j)$ . We denote the fundamental weights of  $H_*$  by  $\eta_1, \dots, \eta_{n-2}$ , where  $\eta_i = \omega_{i+1}|_{T_{H_*}}$  for  $i = 1, \dots, n-2$ .

**Lemma 5.2.** *Let  $\mu = k\omega_{n-1}$ . Let  $v_\mu \in V_\mu^L$  be a highest weight vector. Then*

$$\pi_\mu^L|_{L_*} = \sum_{j=0}^k \pi_{j\eta_{n-2}}^{L_*}$$

and the highest weight vectors are given by  $E_{n,1}^j v_\mu$ . The dimension of the representation of  $L_*$  of highest weight  $j\eta_{n-2}$  is given by  $\binom{n+j-2}{j}$ .

PROOF. Since  $E_{n,1}$  commutes with the root vectors of the positive roots of  $H_*$  the element  $E_{n,1}^j v_\mu$  is a highest weight vector of  $H_*$  of weight  $(k-j)\omega_{n-2}$  unless it is zero. The representation space of  $\pi_{j\eta_{n-2}}^{H_*}$  is isomorphic to  $S^j(\mathbb{C}^{n-1})$  which is of dimension  $\binom{n-2+j}{j}$ .  $\square$

The set  $B(k\omega_{n-1})$  is given by the tuples  $(r, s) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$  such that  $r_j + s_{j+1} = 0$  for  $j = 1, \dots, n-1$ ,  $r_{n-1} + s_n = k$  and  $r_n + s_1 = 0$ . We obtain

$$B(k\omega_{n-1}) = \{\nu_j = (k-j)\varpi_{n-1} + j\varpi_n | j = 0, \dots, k\}.$$

To find  $\Psi_0^{k\omega_{n-1}}(a)$  with  $a \in A$  we invoke Lemma 5.1 and [17, Prop.2.3]. Let  $(\nu_j, k\omega_{n-1})$  be determined by  $(r, s) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ . This pair determines an element  $(\nu_j, \tilde{\mu}^j) \in \widehat{\Lambda}(G, B_{\text{GL}(\mathbb{C}, n)})$ , with  $\tilde{\mu}^j = k\varpi_{n-1} - s_n\varpi_n$ . The bottom of the  $\tilde{\mu}^j$ -well for the pair  $(\text{SL}(n+1, \mathbb{C}), \text{GL}(n, \mathbb{C}))$  is given by

$$B(\tilde{\mu}^j) = \{-p\varpi_1 + (k-j-p)\varpi_{n-1} + (j+p)\varpi_n | p = -j, \dots, -1\} \cup \\ \{(k-j-p)\varpi_{n-1} + (j+2p)\varpi_n | p = 0, \dots, p\},$$

see e.g. [17, Prop.2.3]. This implies that the  $M_{\mathbb{C}}$ -types (see Remark 2.5) are given by  $\frac{j+2p}{2}(\varpi_1 - \varpi_n) + (k-j-p)\eta_{n-2}$ . This observation allows us to refine Lemma 5.2.

**Lemma 5.3.** *Let  $\mu = k\omega_{n-1}$ . Then there exists  $m_1, \dots, m_{k+1} \in \mathbb{Z}$  such that*

$$\Psi_0^{k\omega_{n-1}}(a_1(t)a_2(s)) = \\ \text{diag}(e^{im_1s}, \dots, e^{im_{k+1}s}) \Psi_0^{k\omega_{n-1}}(a_1(t)) \text{diag}(1, e^{i(n-1)s}, \dots, e^{i(n-1)ks}).$$

PROOF. The considerations preceding the lemma indicates what the  $M_{\mathbb{C}}$ -types are and hence with which characters the torus  $A_2$  acts on the  $L_*$ -isotypical components of  $V_\mu^L$ . This allows us to calculate the integers of Lemma 5.2 and the result follows.  $\square$  Our algorithm [16] calculates  $\Psi_0^\mu|_{A_1}$  for  $\mu = k\omega_{n-1}$ . For the values  $(n, k) = (3, 2), (4, 2)$  and  $(5, 2)$  we find

$$(5.1) \quad \Psi_0^{2\omega_{n-1}}(a_1(t)) = \begin{pmatrix} \cos^2(t) & \cos(t) & 1 \\ \cos(t) & \frac{1}{2}(1 + \cos^2(t)) & \cos(t) \\ 1 & \cos(t) & \cos^2(t) \end{pmatrix}.$$



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